Name:______________________________

MA 8663 Section 01 | *Practice Exam 1* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Prove the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

(HINT: The Fourier series for the L^2 function $f(x) = x^2$ on $[-\pi, \pi]$ may help.)

2. Let X be a seperable Hilbert space with ON basis $\{x_k\}$. Let $A : X \to X$ be a bounded linear operator. Prove that

$$\lim_{k\to\infty} |\langle Ax, x_k\rangle| = 0$$

3. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Define the following space

$$H^2(\mathbb{D}) = \{f(z) \text{ is holomorphic on } \mathbb{D} : \|f\| < \infty\}$$

where

$$\|f\|^2 = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

and $z = re^{i\theta}$. Prove that $H^2(\mathbb{D})$ is a Banach space. Be sure to show $\|\cdot\|$ actually defines a norm. Is it a Hilbert space?

4. Let $w \in L^2(\mathbb{R})$ be a weight function. Prove that $L^2(\mathbb{R}, d\mu)$ where $d\mu = w(x)\, dx$ is Hilbert space with the inner product defined as

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\mu$$

Be sure to show that $\langle \cdot, \cdot \rangle$ actually defines an inner product.

5. Let $f \in L^2[0,1]$ and define the following operator $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tf)(s) = \int_0^1 e^{-st} f(t) \, dt$$

1.) Prove that T is a linear bounded.

2.) Prove that T is a bounded operator and $\|T\| \leq 1$

6. Let $f \in L^2[a,b]$ and $g \in L^\infty[a,b]$ and define the following operator $M_g : L^2[a,b] \to L^2[a,b]$ by

$$(M_g f)(x) = g(x)f(x)$$

Prove that M_g is a bounded operator and $\|M_g\| = \|g\|_\infty$.

7. Let $x \in \ell^2(\mathbb{N})$. Define the following operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by:

$$Ax = \left\{k^2(x_{k+1} - x_k)\right\}_{k=1}^{\infty}$$

1.) Show that A is an unbounded operator and compute $\text{Ker}(A)$.

2.) If $x_1 = 0$, compute A^{-1}.

3.) What condition is needed to guarantee that A^{-1} is bounded.

8. Define the following operator on $L^2[0,1]$:

$$A = i\frac{d}{dx}$$

1.) Show that A is an unbounded operator and compute $\text{Ker}(A)$.

2.) If $\text{dom}(A) = \{f \in AC[0,1] \subset L^2[0,1] : f(0) = f(1) = 0\}$, Show that $A^* = A$.

9. Let X be a normed vector space and $T : X \to X$ be a bounded linear operator such that $\|T\| < 1$. Prove that $I - T$ is invertible,

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \text{ and } \left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}$$